MONODROMY FILTRATION AND POSITIVITY

Morihiko Saito

RIMS Kyoto University, Kyoto 606-8502 Japan

Abstract. We study Deligne's conjecture on the monodromy weight filtration on the nearby cycles in the mixed characteristic case, and reduce it to the nondegeneracy of certain pairings in the semistable case. We also prove a related conjecture of Rapoport and Zink which uses only the image of the Cech restriction morphism, if Deligne's conjecture holds for a general hyperplane section. In general we show that Deligne's conjecture is true if the standard conjectures hold.

Introduction

Let $f : X \to S$ be a projective morphism of complex manifolds with relative dimension n where S is an open disk and f is semistable (i.e. $X_0 := f^{-1}(0)$ is a reduced divisor with normal crossings whose irreducible components are smooth). J. Steenbrink [29] constructed a limit mixed Hodge structure by using a resolution of the nearby cycle sheaf on which the monodromy weight filtration can be defined. This limit mixed Hodge structure coincides with the one obtained by W. Schmid [26] using the SL_2-orbit theorem, because the weight filtration coincides with the (shifted) monodromy filtration, see [23] and also [5], [9], [11], [24, 2.3], etc.

A similar construction was then given by M. Rapoport and T. Zink [21] in the case X is projective and semistable over a henselian discrete valuation ring R of mixed characteristic. Here we may assume f semistable because the non semistable case can be reduced to the semistable case by [2] replacing the discrete valuation field with a finite extension if necessary. Then Deligne's conjecture on the monodromy filtration [3, I] is stated as

0.1. Conjecture. The obtained weight filtration coincides with the monodromy filtration shifted by the degree of cohomology.

This conjecture is proved so far in the case $n \le 2$ by [21] (using [2]) and in some other cases (see e.g. [8], [25]). We cannot apply the same argument as in the complex analytic case, because we do not have a good notion of positivity for l-adic sheaves. In the positive characteristic case, however, the conjecture was proved by Deligne [4] (assuming f is the

1991 *Mathematics Subject Classification.* 14C25, 14F20.

Key words and phrases. monodromy filtration, polarization, standard conjecture.

Date: Sept. 22, 2003, v.11

base change of a morphism to a smooth curve over a finite field). In this paper we apply the arguments in [23], [24] to the mixed characteristic case, and show that Conjecture (0.1) is closely related to the standard conjectures which would give a notion of positivity for the pairings of correspondences (but not for the pairings of cohomology groups in general).

Let $Y^{(i)}$ denote the disjoint union of the intersections of i irreducible components of the special fiber X_0. Let k be the residue field of R which is assumed to be a finite field. Let $\overline{k}$ be an algebraic closure of k, and put $Y^{(i)}_{\overline{k}} = Y^{(i)} \otimes_k \overline{k}$. Let l be a prime number different from the characteristic of k. Then the E_1-term of the weight spectral sequence of Rapoport and Zink is given by direct sums of the l-adic cohomology groups of $Y^{(i)}_{\overline{k}}$ which are Tate-twisted appropriately. Its differential d_1 consists of the Cech restriction morphisms and the co-Cech Gysin morphisms which are denoted respectively by ρ and γ in this paper. The primitive cohomology of $Y^{(i)}_{\overline{k}}$ has a canonical pairing induced by Poincaré duality together with the hard Lefschetz theorem [4] (choosing and fixing an ample divisor class of f).

0.2. Theorem. *Conjecture* (0.1) *is true if the restrictions of the canonical pairing to the intersections of the primitive part with* $\operatorname{Im}\rho$ *and with* $\operatorname{Im}\gamma$ *are both nondegenerate.*

So the problem is reduced to the study of the canonical pairing on the primitive part. There are some examples satisfying the assumption of (0.2), see (2.8). If $n = 3$, the converse of (0.2) is also true and the hypothesis on $\operatorname{Im}\gamma$ is always satisfied in this case (using [2], [21]). Note that (0.1) may apparently depend on the choice of l, and we can prove (0.1) for certain l in a simple case where every eigenvalue of the Frobenius action has multiplicity 1, see (2.7) and (5.4). (In this case we can determine the endomorphism ring of a simple motive, see (5.2).) However, the general case seems to be related closely to the standard conjectures.

A conjecture similar to (0.2) but using only the restriction morphisms was noted by Rapoport and Zink in the introduction of [21]. We can prove this conjecture under an inductive hypothesis as follows:

0.3. Theorem. *Assume that Conjecture* (0.1) *holds for a general hyperplane section of the generic fiber, and that the restriction of the modified pairing in* [21] *to* $\operatorname{Im}\rho$ *or the restriction of the canonical pairing to the intersection of the primitive part with* $\operatorname{Im}\rho$ *is nondegenerate. Then* (0.1) *is true.*

In the complex analytic case, the hypothesis of (0.2) is trivially satisfied because of the positivity of polarizations of Hodge structures. We can argue similarly if we have a kind of "positivity" in characteristic $p > 0$. The positivity for a zero-dimensional variety (or a motive) is clear, because the pairing is defined over the subfield $\mathbb{Q}$ of $\mathbb{Q}_l$). In the one-dimensional case, this notion is provided by the theory of Riemann forms for abelian varieties [31] (see also [17], [20]) combined with work of Deligne [6] on the compatibility of the Weil pairing and Poincaré duality (see also (3.4) below). These were used in an essential way for the proof of Conjecture (0.1) for $n = 2$ in [21]. However, for higher dimensional varieties, we do not know any notion of positivity except the standard conjecture of Hodge index type [14]. Using the theory on the standard conjectures (loc. cit.) we show

0.4. Theorem. *Assume that the standard conjectures hold for $Y^{(i)}$, $Y^{(i)}\times_k Y^{(i)}$, and the numerical equivalence and the homological equivalence coincide for $Y^{(i)}\times_k Y^{(i+1)}$ $(i>0)$. Then Conjecture* (0.1) *is true.*

I am quite recently informed that if the generic fiber can be uniformized by the Drinfeld upper half space, Conjecture (0.1) is proved by T. Ito using Theorem (0.2).

Part of this work was done during my stay at the University of Leiden. I thank Professor J. Murre for useful discussions, and the staff of the institute for the hospitality. I also thank Takeshi Saito for a useful discussion about the consequence of de Jong's theory of alternations.

In Sect. 1 and Sect. 2, we review the theory of graded or bigraded modules of Lefschetz-type and prove (0.2) and (0.3). In Sect. 3, we review the work of Deligne on the compatibility of the Weil pairing and Poincaré duality. In Sect. 4, we review the standard conjectures and prove (0.4). In Sect. 5, we study the Frobenius action, and prove (0.1) in some simple cases.

1. Graded Modules of Lefschetz-Type

We first review a theory of morphisms of degree 1 between graded modules of Lefschetz-type [23]. A typical example is given by the restriction or Gysin morphism associated to a morphism of smooth projective varieties whose relative dimension is -1.

1.1. In this and the next sections we denote by Λ a field. By a graded $\Lambda[L]$-module we will mean a finite dimensional graded vector space $M^\bullet$ over Λ having an action of L with degree 2. We call $M^\bullet$ *n-symmetric* if

$$L^j : M^{n-j} \xrightarrow{\sim} M^{n+j} \quad \text{for } j>0.$$

Here the Tate twists are omitted to simplify the notation. We say that $M^\bullet$ is a graded module of Lefschetz-type if it is a 0-symmetric graded $\Lambda[L]$-module. Then we have the Lefschetz decomposition

$$M^j = \bigoplus\nolimits_{i\ge 0} L^i{}_0M^{j-2i} \quad \text{with } {}_0M^{-j} = \operatorname{Ker} L^{j+1} \subset M^{-j}.$$

We say that $f : M^\bullet \to N^\bullet$ is a morphism of degree m between graded modules of Lefschetz-type, if $f(M^j)\subset N^{j+m}$ and f is $\Lambda[L]$-linear. Shifting the degree of $N^\bullet$, it is identified with a graded morphism of a 0-symmetric graded $\Lambda[L]$-module to a $(-m)$-symmetric graded $\Lambda[L]$-module. Considering the image of the primitive part, we see that a morphism of degree 0 preserves the Lefschetz decomposition and there is no nontrivial morphism of negative degree (i.e. a morphism of a 0-symmetric module to an n-symmetric module for $n>0$ is trivial). For a morphism of degree 1, we see that

$$f({}_0M^{-j}) \subset {}_0N^{-j+1} + L\,{}_0N^{-j-1}. \tag{1.1.1}$$

Let $f : M^\bullet \to N^\bullet$ be a morphism of degree 1 between graded modules of Lefschetz-type. We define

$$\operatorname{Im}^0 f = \bigoplus\nolimits_{j\in\mathbb{Z}}(\bigoplus\nolimits_{i\ge 0} L^i(\operatorname{Im} f \cap {}_0N^{-j})), \quad \operatorname{Im}^1 f = \operatorname{Im} f/\operatorname{Im}^0 f.$$

Then $\mathrm{Im}^0 f$ is 0-symmetric, and its primitive part is given by $\mathrm{Im}\, f \cap {}_0N^{-j}$. It has been remarked by T. Ito that the proof of the next lemma easily follows from the definition itself.

1.2. Lemma. *The quotient module* $\mathrm{Im}^1 f$ *is* 1*-symmetric, i.e. we have the bijectivity of*

$$L^{j+1} : (\mathrm{Im}^1 f)^{-j} \to (\mathrm{Im}^1 f)^{j+2} \quad \text{for } j \ge 0, \tag{1.2.1}$$

where the upper index $-j$ *means the degree* $-j$ *part, etc.*

Proof. The surjectivity follows from the 0-symmetry of M. To show the injectivity, let $n = \sum_{i\ge 0} L^i n_i \in (\mathrm{Im}\, f)^{-j}$ with $n_i \in {}_0N^{-j-2i}$, and assume $L^{j+1}n \in (\mathrm{Im}^0 f)^{j+2}$. Since $L^{j+1}n = \sum_{i>0} L^{j+i+1}n_i$ and $\mathrm{Im}^0 f$ is 0-symmetric, we may assume $n_i = 0$ for $i > 0$ by modifying n modulo $\mathrm{Im}^0 f$ if necessary. Then $n \in \mathrm{Im}^0 f$ by definition, and the assertion follows.

1.3. Proposition. *Let* $M^\bullet, N^\bullet$ *be graded* $\Lambda[L]$*-modules of Lefschetz-type, and*

$$f : M^\bullet \to N^\bullet, \quad g : N^\bullet \to M^\bullet$$

be morphisms of degree one. Assume there are nondegenerate pairings of Λ*-modules*

$$\Phi_M : M^j \otimes_\Lambda M^{-j} \to \Lambda, \quad \Phi_N : N^j \otimes_\Lambda N^{-j} \to \Lambda,$$

such that $\Phi_N\circ(f\otimes id)$ *coincides with* $\Phi_M\circ(id\otimes g)$ *up to a nonzero multiple constant and that* $\Phi_M\circ(id\otimes L) = \Phi_M\circ(L\otimes id)$ (*and the same for* Φ_M). *Then*

$$\dim(\mathrm{Im}^0 f)^j = \dim(\mathrm{Im}^1 g)^{j+1}, \quad \dim(\mathrm{Im}^0 g)^j = \dim(\mathrm{Im}^1 f)^{j+1}.$$

Proof. Let a_j, b_j, c_j, d_j denote respectively the above dimensions so that $a_j = a_{-j}$, $b_j = b_{-j}$, etc. By the duality of f and g, we have

$$a_{j+1} + d_j = \dim(\mathrm{Im}\, f)^{j+1} = \dim(\mathrm{Im}\, g)^{-j} = b_{-j-1} + c_{-j} = b_{j+1} + c_j.$$

If we put $p_j = a_j - b_j$, $q_j = c_j - d_j$, we get $p_{j+1} = q_j$ for any $j \in \mathbb{Z}$. So $p_j = q_j = 0$ by the symmetry of p_j, q_j, and the assertion follows.

1.4. Lemma. *With the notation and assumption of* (1.3), *assume that* $n \in \mathrm{Im}\, f \cap {}_0N^{-j}$ *vanishes if* $\Phi_N(f(m), L^j n) = 0$ *for any* $m \in M^{-j-1}$. *Then*

$$\mathrm{Ker}\, g \cap \mathrm{Im}^0 f = 0. \tag{1.4.1}$$

Proof. Let $n = \sum_{0\le i\le r} L^i n_i$ with $n_i \in {}_0N^{-j-2i}$, and assume $g(n) = 0$. Since

$$L^{j+2r} g(n_r) = L^{j+r} g(n) = 0,$$

the assertion is reduced by induction on r to the injectivity of

$$g\circ L^j : \mathrm{Im}\, f \cap {}_0N^{-j} \to M^{j+1}.$$

But this injectivity is clear, because $\Phi_N(f(m), L^j n)$ is equal to $\Phi_M(m, g(L^j n))$ up to a nonzero multiple constant. So the assertion follows.

1.5. Proposition. *With the notation and assumption of* (1.3), *assume the vanishing of* fgf *and the injectivity of*

$$f : \mathrm{Im}^0 g \to N^\bullet, \quad g : \mathrm{Im}^0 f \to M^\bullet. \tag{1.5.1}$$

Then the compositions

$$\mathrm{Im}^0 g \xrightarrow{f} \mathrm{Im}\, f \to \mathrm{Im}^1 f, \quad \mathrm{Im}^0 f \xrightarrow{g} \mathrm{Im}\, g \to \mathrm{Im}^1 g \tag{1.5.2}$$

are isomorphisms, and we have canonical decompositions

$$\mathrm{Im}\, f = \mathrm{Im}^0 f \oplus \mathrm{Im}^1 f, \quad \mathrm{Im}\, g = \mathrm{Im}^0 g \oplus \mathrm{Im}^1 g, \tag{1.5.3}$$

such that the restriction of g *to* $\mathrm{Im}^0 f$ *is injective, and that to* $\mathrm{Im}^1 f$ *is zero, and similarly for the restriction of* f. *Furthermore, we have canonical isomorphisms*

$$\mathrm{Im}\, fg = \mathrm{Im}^1 f, \quad \mathrm{Im}\, gf = \mathrm{Im}^1 g. \tag{1.5.4}$$

Proof. The hypothesis implies $f(\mathrm{Im}^0 g) \cap \mathrm{Im}^0 f = g(\mathrm{Im}^0 f) \cap \mathrm{Im}^0 g = 0$, because the vanishing of fgf is equivalent to that of gfg by duality. So the assertion follows from (1.3).

1.6. Proposition. *With the notation and the assumptions of* (1.3), *assume further that* (1.4.1) *holds,* $\mathrm{Im}\, fg$ *is* 1*-symmetric,* $gfg = 0$, *and*

$$(\mathrm{Ker}\, g \cap \mathrm{Im}\, f)^j = (\mathrm{Im}\, fg)^j \quad \textit{for } j \ne 0. \tag{1.6.1}$$

Then (1.6.1) *holds also for* $j = 0$.

Proof. Since $\mathrm{Ker}\, g \cap \mathrm{Im}\, f \supset \mathrm{Im}\, fg$, it is enough to show the coincidence of the images of both sides of (1.6.1) in $\mathrm{Im}^1 f$ using (1.4.1). We can identify

$$(\mathrm{Ker}\, g \cap \mathrm{Im}\, f)/\mathrm{Im}\, fg \tag{1.6.2}$$

with a graded $\Lambda[L]$-submodule of $\mathrm{Im}^1 f/\mathrm{Im}\, fg$, and the last module is 1-symmetric by hypothesis and (1.2). Furthermore (1.6.2) vanishes except for the degree 0 by hypothesis. So it vanishes at any degree, and the assertion follows.

2. Bigraded Modules of Lefschetz-Type

We prove (0.2–3) after reviewing a theory of bigraded modules of Lefschetz-type [23], [24]. These modules appear in the E_1-term of the Steenbrink-type spectral sequence associated to a semi-stable degeneration.

2.1. Let $M^{\bullet,\bullet}$ be a bigraded $\Lambda[N,L]$-module of Lefschetz-type, i.e. it is a finite dimensional bigraded vector space over Λ having commuting actions of N, L with bidegrees $(2,0)$ and (0.2) respectively such that

$$N^i : M^{-i,j} \xrightarrow{\sim} M^{i,j}\,(i>0),\quad L^j : M^{i,-j} \xrightarrow{\sim} M^{i,j}\,(j>0).$$

Put $M_i^j = M^{-i,j}$, and ${}_{(0)}M_i^j = \operatorname{Ker} N^{i+1} \subset M_i^j$ for $i \ge 0$, and 0 otherwise. Then we have the Lefschetz decomposition for the first index:

$$M_i^j = \textstyle\bigoplus_{a\ge 0} N^a\,{}_{(0)}M_{i+2a}^j. \tag{2.1.1}$$

We define $C_i^j = {}_{(0)}M_i^j$ and

$$C_{i,a}^j = N^a C_{i+a}^j \subset M_{i-a}^j,$$

so that $C_{i,a}^j = 0$ for $i<0$ or $a<0$, and $C_{i,0}^j = C_i^j$. Then

$$N^{i-a} : C_{i,a}^j \xrightarrow{\sim} C_{a,i}^j\,(i>a),\quad L^j : C_{i,a}^{-j} \xrightarrow{\sim} C_{i,a}^j\,(j>0). \tag{2.1.2}$$

Let d be a differential of bidegree $(1,1)$ on $M^{\bullet,\bullet}$ which commutes with N, L and satisfies $d^2=0$. By (1.1.1) applied to the action of N, we have a decomposition $d = d' + d''$ such that

$$d' : C_{i,a}^j \to C_{i-1,a}^{j+1},\quad d'' : C_{i,a}^j \to C_{i,a+1}^{j+1}$$

are differentials which anti-commute with each other. Let $C_i^\bullet = \bigoplus_{j\in\mathbb{Z}} C_i^j$. We have morphisms of degree 1 between graded $\Lambda[L]$-modules

$$\gamma_i : C_i^\bullet \to C_{i-1}^\bullet\ (i>0),\quad \rho_i : C_i^\bullet \to C_{i+1}^\bullet\ (i\ge 0)$$

such that d', d'' are identified with γ_{i-a}, ρ_{i-a} respectively.

We define $H_i^j = Z_i^j/B_i^j$ with

$$Z_i^j = \operatorname{Ker}(d : M_i^j \to M_{i-1}^{j+1}),\quad B_i^j = \operatorname{Im}(d : M_{i+1}^{j-1} \to M_i^j).$$

Then we have the induced morphism $N : H_i^j \to H_{i-2}^j$, and similarly for Z_i^j, B_i^j. For a positive integer i and an integer j, we will consider the condition for the bijectivity of the morphism

$$N^i : H_i^j \to H_{-i}^j \quad \text{for } i>0. \tag{2.1.3}$$

For this we will assume that the action of L induces bijections

$$L^j : H_i^{-j} \xrightarrow{\sim} H_i^j \quad \text{for } j > 0, \tag{2.1.4}$$

and that both terms of (2.1.3) have the same dimension (so that bijectivity is equivalent to injectivity and to surjectivity). Using (2.1.4), the last assumption is satisfied if we have a self-duality of $M^{\bullet,\bullet}$.

Since $M_{i-1}^{\bullet} = C_{i-1}^{\bullet} \oplus NM_{i+1}^{\bullet}$ and $N^i : M_i^j \xrightarrow{\sim} M_{-i}^j\,(i > 0)$, we have a morphism

$$\widetilde{d} := d \circ N^{-i} : Z_{-i}^j \to C_{i-1}^{j+1} \quad \text{for } i > 0.$$

2.2. Proposition. (i) *The surjectivity of* (2.1.3) *is equivalent to*

$$\widetilde{d}(Z_{-i}^j) = (\operatorname{Im} \gamma_i \rho_{i-1})^{j+1}. \tag{2.2.1}$$

(ii) *If* (2.1.3) *for* $(i+2, j)$ *is surjective, then the surjectivity of* (2.1.3) *is further equivalent to*

$$\gamma_i(\operatorname{Ker} \rho_i)^j = (\operatorname{Im} \gamma_i \rho_{i-1})^{j+1}. \tag{2.2.2}$$

(iii) *If furthermore* (2.1.3) *for* $(i+2, j)$ *is surjective and* (2.1.3) *for* $(i+1, j+1)$ *is injective, then the surjectivity of* (2.1.3) *is further equivalent to*

$$(\operatorname{Ker} \rho_{i-1} \cap \operatorname{Im} \gamma_i)^{j+1} = (\operatorname{Im} \gamma_i \rho_{i-1})^{j+1}. \tag{2.2.3}$$

(iv) *If* (2.1.3) *for* $(i+1, j-1)$ *is surjective, then the injectivity of* (2.1.3) *is equivalent to*

$$(\operatorname{Ker} \gamma_i \cap \operatorname{Im} \rho_{i-1})^j = (\operatorname{Im} \rho_{i-1} \gamma_i)^j \;(= (\operatorname{Im} \gamma_{i+1} \rho_i)^j). \tag{2.2.4}$$

Proof. We have by definition $\widetilde{d}(B_{-i}^j) = (\operatorname{Im} \gamma_i \rho_{i-1})^{j+1}$. Then the first assertion is clear because the surjectivity is equivalent to $Z_{-i}^j = B_{-i}^j + N^i Z_i^j$.

For the second, take $m = \sum_{a \ge 0} N^a m_a \in N^{-i} Z_{-i}^j$ with $m_a \in C_{i+2a}^j$ to calculate $\widetilde{d}(Z_{-i}^j)$. We may assume that $m_0 \in \operatorname{Ker} \rho_i$ applying (2.2.1) for $i+2$ to $\sum_{a \ge 1} N^{a-1} m_a$ and modifying m by an element of $d(C_{i+1}^{j-1})$ because this does not change dm. So the second assertion is clear.

Before showing the third assertion, we see that

$$(\operatorname{Ker} \gamma_i \cap \operatorname{Im} \rho_{i-1})^j / \widetilde{d}(Z_{-i-1}^{j-1}) \xrightarrow{\sim} (N^{-i} B_{-i}^j \cap Z_i^j)/B_i^j,$$

because $N^{-i} B_{-i}^j = (\operatorname{Im} \rho_{i-1})^j + B_i^j$. Then the last assertion follows from (2.2.1).

For the third assertion, take $\gamma_i m \in (\operatorname{Ker} \rho_{i-1})^{j+1}$, where $m \in C_i^j$ with $\rho_{i-1} \gamma_i m = 0$. Then $\rho_i m \in \operatorname{Ker} \gamma_{i+1}$, and we may assume $\rho_i m = 0$ using (2.2.4) for $(i+1, j+1)$ and

modifying m by $\gamma_{i-1}m'$ because it does not change $\gamma_i m$. So the assertion follows from (2.2.2).

2.3. Application. With the notation of the introduction we may assume

$$C_i^j = H^{n-i+j}(Y_{\overline{k}}^{(i+1)}, \mathbb{Q}_l(-i)), \quad H_i^j = \mathrm{Gr}^W_{n+j+i} H^{n+j}(X_{\overline{K}}, \mathbb{Q}_l), \tag{2.3.1}$$

and ρ_i, γ_{i+1} are respectively the Cech restriction and co-Cech Gysin morphisms, which are dual of each other up to a sign. In particular, $\rho_j\gamma_{j+1}\rho_j = -\gamma_{j+2}\rho_{j+1}\rho_j = 0$, i.e. the first hypothesis of Proposition (1.5) is satisfied. Note that N is the logarithm of the monodromy, and L is given by the ample divisor class of f, see [11], [21]. So the bijectivity of (2.1.4) follows from the hard Lefschetz theorem for the generic fiber together with the strict compatibility of the weight filtration [4].

2.4. Conjecture of Rapoport and Zink. For a smooth projective $\overline{k}$-variety Y with an ample line bundle L, we have a canonical pairing on $H^j(Y, \mathbb{Q}_l)$ by Poincaré duality and the Hard Lefschetz theorem [4]. Using further the Lefschetz decomposition and modifying the sign as in (4.1), we get a modified pairing on $H^j(Y, \mathbb{Q}_l)$ as in [21]. Rapoport and Zink noted there that Conjecture (0.1) would be verified if the restriction of this modified pairing to $\mathrm{Im}\,\rho$ is nondegenerate. Note that the hypothesis of (1.4) is satisfied under the above hypothesis (where $f = \rho_{i-1}$ and $g = \gamma_i$) because the modified pairing coincides with the canonical pairing up to a sign if one factor belongs to the primitive part.

2.5. Proof of (0.2) and (0.3). By [21], the E_1-term of the weight spectral sequence has a structure of bigraded $\mathbb{Q}_l$-modules of Lefschetz-type, see (2.3). Then (2.2.3–4) follows from (1.4–5), and Theorem (0.2) follows.

For Theorem (0.3) we will show by decreasing induction on i that (2.1.3) is bijective (or equivalently, injective) and $\mathrm{Im}\,\gamma_i\rho_{i-1}$ is 1-symmetric. Assume the two assertions are true for $i+r$ with $r > 0$. Considering the restriction morphism to a general hyperplane section of the generic fiber and using the weak Lefschetz theorem, we see that (2.1.3) is injective for $j < 0$, because the restriction morphism is strictly compatible with the weight filtration. Then (2.1.3) is injective also for $j > 0$ by (2.1.4), and it is injective for any j by (2.2) (iv) and (1.6) where $f = \rho_{i-1}$ and $g = \gamma_i$. So it remains to show that $\mathrm{Im}\,\gamma_i\rho_{i-1}$ is 1-symmetric.

Since the injectivity of the first morphism of (1.5.1) follows from (1.4), the assertion is reduced by (1.5) to the injectivity of $\rho_{i-1} : \mathrm{Im}^0\gamma_i \to C_i^\bullet[1]$, i.e. to the vanishing of

$$\mathrm{Ker}(\rho_{i-1} : \mathrm{Im}^0\gamma_i \to (\mathrm{Im}\,\rho_{i-1}\gamma_i)[1]). \tag{2.5.1}$$

Here $[m]$ means a shift of degree for an integer m, i.e. $(M[m])^i = M^{i+m}$. We see that (2.5.1) is 0-symmetric, because $\mathrm{Im}^0\gamma_i$ and $(\mathrm{Im}\,\rho_{i-1}\gamma_i)[1]$ are (using inductive hypothesis). Furthermore (2.5.1) is a submodule of $\mathrm{Ker}\,\rho_{i-1} \cap \mathrm{Im}\,\gamma_i$, and the latter is identified by (2.2.3–4) with

$$\mathrm{Im}(\gamma_i : (\mathrm{Im}\,\rho_{i-1})[-1] \to \mathrm{Im}\,\gamma_i) = (\mathrm{Im}\,\rho_{i-1}/\mathrm{Im}\,\rho_{i-1}\gamma_i)[-1].$$

The last module is an extension of a 2-symmetric module by a 1-symmetric module, because $\mathrm{Im}\,\rho_{i-1}\gamma_i \cap \mathrm{Im}^0\rho_{i-1} = 0$ by (1.4) and $\mathrm{Im}\,\rho_{i-1}\gamma_i\,(= \mathrm{Im}\,\gamma_{i+1}\rho_i)$ is 1-symmetric by

inductive hypothesis. But there is no nontrivial morphism of a 0-symmetric module to an n-symmetric module for $n > 0$, see (1.1). So the assertion follows.

2.6. First cohomology case. The restriction of the pairing to the image of ρ_{i-1} in $H^1(Y_{\overline{k}}^{(i+1)}, \mathbb{Q}_l)$ is always nondegenerate by the abelian-positivity in Sect. 3, because ρ_{i-1} is associated to the morphism of Picard varieties using the Tate module.

2.7. Three-dimensional case. Assume $n = 3$. Then we can show that Conjecture (0.1) is equivalent to the nondegeneracy of the restrictions of the canonical pairing to

$$H^3(Y_{\overline{k}}^{(1)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\gamma, \quad H^2(Y_{\overline{k}}^{(2)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\rho. \tag{2.7.1}$$

Indeed, using the abelian-positivity in Sect. 3, the arguments in (2.5) show that these conditions in this case are equivalent respectively to the bijectivity of

$$N : H_1^{-1} \to H_{-1}^{-1}, \quad N : H_1^0 \to H_{-1}^0.$$

Since the first morphism is bijective by using a general hyperplane section, the nondegeneracy of the pairing for the intersection with $\operatorname{Im}\gamma$ in (0.2) is always true for $n = 3$. Note that the bijectivity of the first morphism cannot be proved by using simply the abelian positivity in Sect. 3 unless we know that $\operatorname{Ker}\gamma \cap H^1(Y_{\overline{k}}^{(2)}, \mathbb{Q}_l)$ corresponds to an abelian subvariety of the Picard variety.

If $\operatorname{Im}\rho \subset H^2(Y_{\overline{k}}^{(2)}, \mathbb{Q}_l)$ is contained in the subspace generated by algebraic cycle classes, then Conjecture (0.1) can be reduced to $D(Y^{(1)})$ or $A(Y^{(1)}, L)$ in the notation of (4.1) (or to the Tate conjecture [30] for divisors on $Y^{(1)}$) because $D(Y^{(2)})$, $D(Y^{(1)})$ for divisors and $I(Y^{(2)}, L)$ are known.

If, for every eigenvalue of the Frobenius action on $\operatorname{Im}\rho \cap H^2(Y_{\overline{k}}^{(2)}, \mathbb{Q}_l)^{\mathrm{prim}}$, its multiplicity as an eigenvalue of the Frobenius action on $H^2(Y_{\overline{k}}^{(2)}, \mathbb{Q}_l)^{\mathrm{prim}}$ is 1, then Conjecture (0.1) can be proved for certain prime numbers l, see (5.4).

2.8. Example. Let $V \to S := \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $n+1$. For $i = 0, \dots, r$, let Z_i be a smooth hypersurface of V defined by a section P_i of a relative ample line bundle L_i. Assume $L_0 = \bigotimes_{1 \le i \le r} L_i$, and $\bigcup_{0 \le i \le r} Z_i$ is a divisor with normal crossings relative to R (i.e. the fiber over $\operatorname{Spec} k$ is a divisor with normal crossings). Let π be a generator of the maximal ideal of R, and define

$$P = P_1 \cdots P_r + P_0 \pi.$$

Let X be the hypersurface of V defined by P. (This is an analogue of [7] for $r = 2$.) Then Conjecture (0.1) is true for a semi-stable model of the generic fiber of X although X is not semistable in general.

Indeed, consider an iteration of blow-ups $\sigma_j : V^{(j)} \to V^{(j-1)}$ along the proper transform of $Z_0 \cap Z_j$ for $j = 1, \dots, r-1$, where $V^{(0)} = V$. We have local coordinates $x_0, \dots, x_n$ over R such that P is locally given by

$$u x_1 \cdots x_m + x_0 \pi,$$

where x_i is the restriction of P_i for $i \le m$ and u is locally invertible. Then the proper transform of P by the blow-up along $Z_0 \cap Z_1$ is locally given by

$$ux_2 \cdots x_m + x'_0\pi \quad \text{or} \quad ux''_1 x_2 \cdots x_m + \pi,$$

where $(x'_0, x_1, x_2, \cdots, x_n)$ and $(x_0, x''_1, x_2, \cdots, x_n)$ are local coordinate systems on open subvarieties U', U'' of $V^{(1)}$ such that $x_0 = x'_0 x_1$ on U' and $x_1 = x''_1 x_0$ on U''. Here the pull-back of x_i is also denoted by x_i to simplify the notation. Since U'' does not intersect the proper transform of $Z_0 \cap Z_j$ for $j > 1$, we can proceed by induction, and get a semi-stable model. Its generic fiber is same as that of X because the intersection of the center of the blow-up with the generic fiber of (the proper transform of) X is a locally principal divisor on the generic fiber. Furthermore the $Y^{(i)}$ are lifted to smooth projective schemes over R so that the assumption of (0.2) is verified by using Hodge theory.

3. Abelian-Positivity

3.1. Canonical pairing. Let A be an abelian variety over a field k. We denote by $A^\vee$ its dual variety, and by $T_l A_{\overline{k}}$ the Tate module of $A_{\overline{k}} := A \otimes_k \overline{k}$ where l is a prime number different form $\operatorname{char} k$. Using the Kummer sequence, we have a canonical isomorphism

$$H^1(A_{\overline{k}}, \mu_n) = A^\vee(\overline{k})_n \; (:= \operatorname{Ker}(n : A^\vee(\overline{k}) \to A^\vee(\overline{k}))), \tag{3.1.1}$$

where n is an integer prime to $\operatorname{char} k$. Then, passing to the limit, we get

$$H^1(A_{\overline{k}}, \mathbb{Z}_l(1)) = T_l A^\vee_{\overline{k}}. \tag{3.1.2}$$

Since the left-hand side of (3.1.1) is identified with

$$\operatorname{Hom}(A(\overline{k})_n, \mu_n),$$

using torsors [6], we get the canonical pairing of Weil [31] (see also [17], [20]):

$$A(\overline{k})_n \otimes A^\vee(\overline{k})_n \to \mu_n, \quad T_l A_{\overline{k}} \otimes_{\mathbb{Z}_l} T_l A^\vee_{\overline{k}} \to \mathbb{Z}_l(1). \tag{3.1.3}$$

To get a pairing of $T_l A_{\overline{k}}$, we take a divisor D on A which induces a morphism

$$\varphi_D : A \to A^\vee, \tag{3.1.4}$$

such that $\varphi_D(a) \in A^\vee(\overline{k})$ for $a \in A(\overline{k})$ is given by $T_a^* D_{\overline{k}} - D_{\overline{k}}$, where T_a is the translation by a (see loc. cit.) Note that φ_D depends only on the algebraic equivalence class of D, and φ_D is an isogeny if D is ample.

If the pairing is induced by an ample divisor on A, its restriction to $T_l B_{\overline{k}}$ for any abelian subvariety B of A is nondegenerate, because we have the commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow{\scriptstyle \varphi_{D|_B}} & & \downarrow{\scriptstyle \varphi_D} \\ B^\vee & \longleftarrow & A^\vee \end{array} \tag{3.1.5}$$

Note that this holds only for subgroups of $T_l A_{\overline{k}}$ corresponding to abelian subvarieties.

We say that a pairing of a $\mathbb{Q}_l$-module V with a continuous action of $G := \mathrm{Gal}(\overline{k}/k)$ is abelian-positive if there exists an abelian variety with an ample divisor D such that V is isomorphic to $T_l A_{\overline{k}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ up to a Tate twist as a $\mathbb{Q}_l[G]$-module and the pairing corresponds to the one on $T_l A_{\overline{k}}$ defined by the canonical pairing and φ_D. Note that abelian-positive pairings (having the same weight) are stable by direct sums.

3.2. Compatibility of the cycle classes. Let A be an abelian variety over k, X a smooth projective variety over k, and D a divisor on $A \times_k X$ such that its restriction to $\{0\} \times X$ is rationally equivalent to zero. Let P be the Picard variety of X. Then D induces a morphism of abelian varieties

$$\Psi_D : A \to P,$$

such $\Psi_D(a) \in P(\overline{k})$ for $a \in A(\overline{k})$ is defined by the restriction of $D_{\overline{k}}$ to $\{a\} \times X_{\overline{k}}$, see [31] (and also [17], [20]).

Let $cl(D)^{1,1} \in H^1(A_{\overline{k}}, R^1(pr_1)_* \mu_n)$ denote the $(1,1)$-component of the cycle class of D, where n is an integer prime to $\mathrm{char}\, k$, and pr_1 is the first projection. Assume $\mathrm{NS}(X)$ is torsion-free. Then $R^1(pr_1)_* \mu_n$ is a constant sheaf on $A_{\overline{k}}$ with fiber $H^1(X_{\overline{k}}, \mu_n) = P(\overline{k})_n$, and we get

$$cl(D)^{1,1} \in H^1(A_{\overline{k}}, R^1(pr_1)_* \mu_n) = \mathrm{Hom}(A(\overline{k})_n, P(\overline{k})_n).$$

3.3. Theorem (Deligne). *The induced morphism $\Psi_D : A(\overline{k})_n \to P(\overline{k})_n$ coincides with $-cl(D)^{1,1}$.*

(The proof is essentially the same as in [6].)

3.4. Corollary (Deligne). *Let C be a smooth projective curve over a field k, and J its Jacobian. Then we have a canonical isomorphism $H^1(C_{\overline{k}}, \mathbb{Z}_l(1)) = T_l J(\overline{k})$ such that Poincaré duality on $H^1(C_{\overline{k}}, \mathbb{Z}_l(1))$ is identified with the pairing of $T_l J(\overline{k})$ given by the canonical pairing* (3.1.3) *together with* (3.1.4) *for the theta divisor on J.*

Proof. We may assume that C has a k-rational point replacing k with a finite extension k' and C with $C \otimes_k k'$ if necessary. Choosing a k-rational point of C, we have a morphism $f : C \to J$. It is well-known that this induces isomorphisms

$$f^* : H^1(J_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^1(C_{\overline{k}}, \mathbb{Z}_l),$$
$$f_* : H^1(C_{\overline{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^{2g-1}(J_{\overline{k}}, \mathbb{Z}_l(g-1)).$$

These are independent of the choice of the k-rational point of C, because a translation on J acts trivially on the cohomology of J. Since f_* and f^* are dual of each other, the canonical pairing on $H^1(C_{\overline{k}}, \mathbb{Z}_l)$ is identified by f^* with the pairing on $H^1(J_{\overline{k}}, \mathbb{Z}_l)$ given by Poincaré duality and $f_* \circ f^*$.

Let $\Gamma_1 = m^*\Theta - pr_1^*\Theta - pr_2^*\Theta$, where Θ is the theta divisor, and $m : J \times_k J \to J$ is the multiplication. Let $\Gamma_2 \in \mathrm{CH}^g(J \times_k J)$ be the diagonal of $f(C) \subset J$ so that $f_* \circ f^*$ is identified with Γ_2. Since the canonical pairing (3.1.3) can be identified with Poincaré

duality (see (3.6)), the assertion is reduced by (3.3) (where $A = X = J$ and $D = \Gamma_1$) to that the actions of the correspondences

$$(\Gamma_1)_* : H^{2g-1}(J_{\overline{k}}, \mathbb{Z}_l) \to H^1(J_{\overline{k}}, \mathbb{Z}_l(1-g)),$$
$$(\Gamma_2)_* : H^1(J_{\overline{k}}, \mathbb{Z}_l(1-g)) \to H^{2g-1}(J_{\overline{k}}, \mathbb{Z}_l)$$

are inverse of each other up to sign, or equivalently, that the action of the composition of $\Gamma_2 \circ \Gamma_1$ on $H^{2g-1}(J_{\overline{k}}, \mathbb{Z}_l)$ is the multiplication by -1. Here it is enough to show the assertion for the action on the Albanese variety of J, see (3.7). For $a, b \in J(\overline{k})$, we see that the image of $[a] - [b]$ by the action of $\Gamma_2 \circ \Gamma_1$ is given by

$$f_* f^*(T_a^* \Theta - T_b^* \Theta). \tag{3.4.1}$$

Let $C^{(j)}$ denote the j-th symmetric power of C. Then f induces $f^{(j)} : C^{(j)} \to J$, and $f^{(g)}$ is birational [31]. So there is a nonempty Zariski-open subset U of J such that for $a \in U(\overline{k})$, there exists uniquely $\{c_1, \dots, c_g\} \in C^{(g)}(\overline{k})$ satisfying

$$-a - \sum_{i \neq j} f(c_i) = f(c_j). \tag{3.4.2}$$

Since $\Theta = f^{(g-1)}(C^{(g-1)})$, it implies that $T_a^* \Theta^- \cap f(C) = \{f(c_j)\}$, where $\Theta^- = (-1)^* \Theta$. But $\varphi_\Theta = \varphi_{\Theta^-}$, because the action of $-1 : J \to J$ on $\mathrm{NS}(J)$ is the identity. So the assertion follows from (3.4.1–2).

3.5. First cohomology case. Let Y be a smooth projective variety with an ample divisor class L. Then the canonical pairing on $H^1(Y_{\overline{k}}, \mathbb{Q}_l)$, defined by Poincaré duality and L, is abelian-positive. Indeed, we may assume L is very ample, and take C a smooth closed subvariety of dimension 1 which is an intersection of general hyperplane sections (replacing k with a finite extension if necessary). Then the composition of the restriction and Gysin morphisms

$$H^1(Y_{\overline{k}}, \mathbb{Q}_l) \to H^1(C_{\overline{k}}, \mathbb{Q}_l) \to H^{2n-1}(Y_{\overline{k}}, \mathbb{Q}_l(n-1))$$

coincides with L^{n-1}, where $n = \dim X$. So the pairing on $H^1(Y_{\overline{k}}, \mathbb{Q}_l)$ is identified with the restriction of the natural pairing on $H^1(C_{\overline{k}}, \mathbb{Q}_l)$ to $H^1(Y_{\overline{k}}, \mathbb{Q}_l)$.

3.6. Complement to the proof of (3.4), I. Let X be an irreducible smooth projective variety over a field k having a k-rational point 0. Let P_X be the Picard variety of X. Then we have a canonical morphism $\mathrm{Alb} : X \to P_X^\vee$ sending 0 to 0 so that $P_X^\vee$ is identified with the Albanese variety Alb_X of X. (Indeed, for an abelian variety A and a morphism $f : (X, 0) \to (A, 0)$, we have $f^\vee : A^\vee \to P_X$ and $f^{\vee\vee} : P_X^\vee \to A$ so that $f^{\vee\vee} \circ \mathrm{Alb} = f$, using the theory of divisorial correspondences.)

Let $n = \dim X$, and l a prime number different from the characteristic of k. Let $V_l M = T_l M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ for an abelian group M. Then, using the Kummer sequence together with the arguments in (3.1), we have canonical isomorphisms

$$V_l P_X(\overline{k}) = H^1(X_{\overline{k}}, \mathbb{Q}_l(1)),$$
$$V_l \mathrm{Alb}_X(\overline{k}) = \mathrm{Hom}(V_l P_X(\overline{k}), \mathbb{Q}_l(1)) = H^{2n-1}(X_{\overline{k}}, \mathbb{Q}_l)(n),$$

where the last isomorphism follows from the first together with Poincaré duality, and the last term is identified with $H_1(X_{\overline{k}},\mathbb{Q}_l):=\mathrm{Hom}(H^1(X_{\overline{k}},\mathbb{Q}_l),\mathbb{Q}_l)$.

3.7. Complement to the proof of (3.4), II. Let X,Y be irreducible smooth projective varieties over a field k having a k-rational point 0. Let $\Gamma\in\mathrm{CH}^m(X\times_kY)$ with $m=\dim Y$. Then Γ induces morphisms

$$(X,0)\to(\mathrm{Alb}_Y,0),\quad \Gamma_*:\mathrm{Alb}_X\to\mathrm{Alb}_Y,$$

and the induced morphism $\Gamma_*:V_l\mathrm{Alb}_X(\overline{k})\to V_l\mathrm{Alb}_Y(\overline{k})$ is identified by (3.6) with

$$\Gamma_*:H^{2n-1}(X_{\overline{k}},\mathbb{Q}_l)(n)\to H^{2m-1}(Y_{\overline{k}},\mathbb{Q}_l)(m),$$

which is given by $cl(\Gamma)^{1,2m-1}\in H^1(X_{\overline{k}},\mathbb{Q}_l)\otimes H^{2m-1}(Y_{\overline{k}},\mathbb{Q}_l)(m)$, the $(1,2m{-}1)$-component of the cycle class of Γ_*, see also [22]. Although this is well-known to specialists, its proof does not seem to be completely trivial, and we give here a short sketch for the convenience of the reader.

First we may replace Y with Alb_Y (by composing Γ with the graph of $Y\to\mathrm{Alb}_Y$) so that the assertion is reduced to the case Y is an abelian variety A. Consider $\mathrm{Alb}(\Gamma)\in\mathrm{CH}^m(X\times_kA)$ which is a graph of a morphism of X to A, and is defined by using the additive structure of A (applied to the restriction of Γ to the generic fiber of $X\times_kA\to X$). Then the assertion is easily verified if Γ is replaced by $\mathrm{Alb}(\Gamma)$, because $\mathrm{Alb}(\Gamma)$ is then extended to an element of $\mathrm{CH}^m(\mathrm{Alb}_X\times_kA)$ which is a graph of a morphism of abelian varieties.

So the assertion is reduced to that $cl(\Gamma)^{1,2m-1}=0$ if $\mathrm{Alb}(\Gamma)=0$. Here we may assume k is algebraically closed. Replacing X with a variety which is étale over X, we may assume that Γ is a linear combination of the graphs of morphisms of X to A (where X is smooth and irreducible, but may be nonproper). By induction on the number of the components of Γ, we may assume that

$$\Gamma=\Gamma_{g_1+g_2}-\Gamma_{g_1}-\Gamma_{g_2}-\Gamma_0$$

for morphisms $g_i:X\to A\,(i=1,2)$, where Γ_{g_i} denotes the graph of g_i.

Let $\widetilde{\Gamma}$ denote the pullback of the cycle $\Gamma_{g_1}-\Gamma_0$ by the projection $A\times_kX\times_kA\to X\times_kA$ sending (a,x,b) to $(x,a+b)$. Then $\Gamma_{g_1}-\Gamma_0$ and $\Gamma_{g_1+g_2}-\Gamma_{g_2}$ coincide with the pull-backs of $\widetilde{\Gamma}$ by the inclusions $X\times_kA\to A\times_kX\times_kA$ sending (x,a) to $(0,x,a)$ and $(-g_2(x),x,a)$ respectively. Since these inclusions are sections of the projection to the second and third factors, it is enough to show that the Künneth component of the cycle class of $\widetilde{\Gamma}$ in

$$H^1(A\times_kX,\mathbb{Q}_l)\otimes H^{2m-1}(A,\mathbb{Q}_l)(m)$$

comes from $H^1(X,\mathbb{Q}_l)\otimes H^{2m-1}(A,\mathbb{Q}_l)(m)$ by $pr_2\times id$, where $pr_2:A\times_kX\to X$ is the second projection. The last assertion is easily verified by using the Künneth decomposition.

4. Standard Conjectures

4.1. Let Y be an equidimensional smooth projective variety over a field k. We fix an ample divisor class L of Y. Then L acts on étale cohomology. Let $n = \dim Y$. By the hard Lefschetz theorem [4], we have

$$L^j : H^{n-j}(Y_{\overline{k}}, \mathbb{Q}_l) \xrightarrow{\sim} H^{n+j}(Y_{\overline{k}}, \mathbb{Q}_l(j)) \;\; (j > 0),$$

which implies the Lefschetz decomposition

$$H^j(Y_{\overline{k}}, \mathbb{Q}_l) = \bigoplus\nolimits_{i \geq 0} L^i H^{j-2i}(Y_{\overline{k}}, \mathbb{Q}_l(-i))^{\mathrm{prim}}.$$

This induces a morphism

$$\Lambda : H^j(Y_{\overline{k}}, \mathbb{Q}_l) \to H^{j-2}(Y_{\overline{k}}, \mathbb{Q}_l(-1)),$$

such that for $m \in H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$, we have $\Lambda(L^i m) = L^{i-1}m$ if $i > 0$, and 0 otherwise. The standard conjecture $B(Y)$ asserts that Λ is algebraically defined as an action of a correspondence.

We define

$$A^j(Y) = \mathrm{Im}(cl : \mathrm{CH}^j(Y)_{\mathbb{Q}} \to H^{2j}(Y_{\overline{k}}, \mathbb{Q}_l(j))), \tag{4.1.1}$$

so that we have the injective morphism $A^j(Y) \to H^{2j}(Y_{\overline{k}}, \mathbb{Q}_l(j))$. Note that $A^j(Y)$ may apparently depend on the choice of l. The standard conjecture $A(Y, L)$ asserts the isomorphism

$$L^{n-2i} : A^i(Y) \xrightarrow{\sim} A^{n-i}(Y) \;\; (0 \leq i < n/2).$$

This is independent of L if the $A^i(Y)$ are finite dimensional, because it is equivalent to the equality of the dimensions by the hard Lefschetz theorem for the étale cohomology groups. The conjecture $A(Y, L)$ follows from $B(Y)$, and implies that the Lefschetz decomposition is compatible with the subspace $A^j(Y)$ so that

$$A^j(Y) = \bigoplus\nolimits_{i \geq 0} L^i A^{j-i}(Y)^{\mathrm{prim}}.$$

The standard conjecture of Hodge index type $I(Y, L)$ asserts that the pairing

$$(-1)^j \langle L^{n-2j} a, b \rangle \quad \text{for } a, b \in A^j(Y)^{\mathrm{prim}}$$

is positive definite for $0 \leq j \leq n/2$.

We will denote by $D(Y)$ the conjecture which asserts the coincidence of the homological equivalence and the numerical equivalences for the cycles on Y (i.e. the canonical pairing between $A^j(Y)$ and $A^{n-j}(Y)$ is nondegenerate for $A^j(Y)$). This is equivalent to $A(Y, L)$ under the assumption $I(Y, L)$, and implies the injectivity of

$$A^j(Y) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to H^{2j}(Y_{\overline{k}}, \mathbb{Q}_l(j)), \tag{4.1.2}$$

and also the independence of $A^j(Y)$ of the choice of l. It is known that $D(Y)$ is true for divisors (by Matsusaka), and $I(Y,L)$ is true for surfaces (by Segre [28]), see [14], [16] (and also [10], [19]).

By the Lefschetz decomposition, we have an isomorphism

$$^* : H^{n+j}(Y_{\overline{k}},\mathbb{Q}_l) \xrightarrow{\sim} H^{n-j}(Y_{\overline{k}},\mathbb{Q}_l(-j))$$

such that for $m \in H^i(Y_{\overline{k}},\mathbb{Q}_l(-a))^{\mathrm{prim}}$, we have

$$^*(L^a m) = (-1)^{i(i+1)/2} L^{n-i-a} m.$$

Combined with Poincaré duality, this induces a pairing on $H^\bullet(Y_{\overline{k}},\mathbb{Q}_l)$ defined by $\langle m, {}^*n\rangle$ for $m,n \in H^j(Y_{\overline{k}},\mathbb{Q}_l)$.

For a nonzero correspondence $\Gamma \in A^n(Y\times_k Y) \subset \mathrm{End}(H^\bullet(Y_{\overline{k}},\mathbb{Q}_l))$, we define Γ' to be the composition of *, ${}^t\Gamma$ and *, where ${}^t\Gamma$ is the transpose of Γ. Then Γ' is algebraic and

$$\mathrm{Tr}(\Gamma'\circ\Gamma) > 0, \tag{4.1.3}$$

if $B(Y)$ and $I(Y\times_k Y, L\otimes 1 + 1\otimes L)$ are satisfied, see [14]. (This is an analogue of the positivity of the Losati involution.) Note that $H^\bullet(Y_{\overline{k}},\mathbb{Q}_l)\otimes H^\bullet(Y_{\overline{k}},\mathbb{Q}_l)$ is the direct sum of

$$S_{i,j} := \sum_{a=0}^{i}\sum_{b=0}^{j} L^a H^{n-i}(Y_{\overline{k}},\mathbb{Q}_l)^{\mathrm{prim}} \otimes L^b H^{n-j}(Y_{\overline{k}},\mathbb{Q}_l)^{\mathrm{prim}}.$$

For $i=j$, the primitive part of $S_{i,i}$ of degree $2n$ for the action of $L\otimes 1 + 1\otimes L$ is isomorphic to $H^{n-i}(Y_{\overline{k}},\mathbb{Q}_l)^{\mathrm{prim}} \otimes H^{n-i}(Y_{\overline{k}},\mathbb{Q}_l)^{\mathrm{prim}}$, and this isomorphism is compatible with the canonical pairing up to a nonzero multiple constant.

If $B(Y)$ and $I(Y\times_k Y, L\otimes 1+1\otimes L)$ are true, then $A^n(Y\times_k Y)$ is a semisimple algebra and there are projectors to primitive parts (and $D(Y\times_k Y)$ holds), see [14], [15], [18]. We have the projector to each cohomology group by the algebraicity of the Künneth components using the Frobenius morphism [13] (because k is assumed to be a finite field). By Jannsen [12] it is actually enough to assume $D(Y\times_k Y)$ for the semisimplicity of $A^n(Y\times_k Y)$.

We will denote by

$$\iota : A^n(Y\times_k Y) \to \mathrm{End}(H^\bullet(Y_{\overline{k}},\mathbb{Q}_l))$$

the canonical injection induced by (4.1.1).

4.2. Remarks. (i) Assume $D(Y\times_k Y)$ holds. Then $A := A^n(Y\times_k Y)$ is a direct product of full matrix algebras over skew fields by [12] (together with Wedderburn's theorem). For an element f of A, there exists an idempotent π such that $\mathrm{Im}\,\iota(\pi) = \mathrm{Im}\,\iota(f)$ in $H^\bullet(Y_{\overline{k}},\mathbb{Q}_l)$. Indeed, using projectors (and replacing $H^\bullet(Y_{\overline{k}},\mathbb{Q}_l)$ with the corresponding subspace V), the assertion is reduced to the case where A is a full matrix algebra over a skew field. Then a matrix can be modified to a simple form by using actions of invertible matrices from both sides as well-known (and a conjugate of an idempotent is an idempotent).

(ii) For a morphism of k-varieties $f : X \to Y$, the idempotent corresponding to the image of f^* is obtained by considering the graph of f and applying the above argument

to the disjoint union of X and Y if $\dim X = \dim Y$. In general we can replace X or Y by a product with $\mathbb{P}^m$ (using the Künneth decomposition for $\mathbb{P}^m$), see [27]. Note that $D(X\times_k Y\times_k \mathbb{P}^m)$ can be reduced to $D(X\times_k Y)$.

4.3. Proof of (0.4). By (4.1) and (4.2) there exists an idempotent π of $A^n(Y\times_k Y)$ such that $\operatorname{Im}\iota(\pi)$ coincides with the intersection of the primitive part with the image of the Cech restriction or co-Cech Gysin morphism.

We show in general that for a projector π of $A^n(Y\times_k Y)$, the restriction of the pairing to $\operatorname{Im}\iota(\pi)$ is nondegenerate if $\operatorname{Im}\iota(\pi)$ is contained in $H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$. For this it is enough to show

$$\operatorname{Im}\iota(\pi)\cap \operatorname{Ker}\iota(\pi') = 0, \tag{4.3.1}$$

where π' is the composition of *, ${}^t\pi$, * as in (4.1.3). We see that (4.3.1) follows from (4.1.3) if π corresponds to a simple motive, because the intersection is defined as a motive (and the forgetful functor associating the underlying vector space commutes with Im, Ker and the intersection). In general, we consider a simple submotive of $\operatorname{Im}\pi$. Let π_0 be the projector defining it. Then (4.3.1) holds for π_0, and we get a decomposition

$$H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}} = \operatorname{Im}\iota(\pi_0)\oplus(\operatorname{Ker}\iota(\pi'_0)\cap H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}}),$$

which is defined motivically, and is compatible with $\operatorname{Im}\iota(\pi)$, i.e.

$$\operatorname{Im}\iota(\pi) = \operatorname{Im}\iota(\pi_0)\oplus(\operatorname{Ker}\iota(\pi'_0)\cap \operatorname{Im}\iota(\pi)).$$

Therefore, replacing $H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ with $\operatorname{Ker}\iota(\pi'_0)\cap H^j(Y_{\overline{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$, we can proceed by induction. This completes the proof of (0.4).

5. Frobenius Action

5.1. Weil Conjecture. Let Y be an equidimensional smooth projective k-variety, and put $A = A^n(Y\times_k Y)$ where $n = \dim Y$. Let $q = |k|$, $p = \operatorname{char} k$. We denote by g the graph of the q-th power Frobenius. Then g belongs to the center of A (using the compatibility with the Galois action). To simplify the relation with the eigenvalues, let $P(T)$ denote the characteristic polynomial of the action of the Frobenius g on $H^\bullet(Y_{\overline{k}}, \mathbb{Q}_l)$ $(:= \bigoplus_{i\in\mathbb{Z}} H^i(Y_{\overline{k}}, \mathbb{Q}_l))$ such that the eigenvalues of the Frobenius action are the roots of $P(T)$ (this normalization is different from the one used in [4], [13], etc.) Then $P(T)$ is a monic polynomial with integral coefficients and is independent of $l\,(\neq p)$. The eigenvalues of the action of g on $H^i(Y_{\overline{k}}, \mathbb{Q}_l)$ are algebraic integers whose image by any embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ has absolute value $q^{i/2}$, see [4].

Let E be an algebraic number field. Consider the decomposition $P(T) = \prod_j P_j(T)^{m_j}$ in $E[T]$ where the $P_j(T)$ are monic irreducible polynomials whose coefficients are algebraic integers in E. Let v be a prime (i.e. a finite place) of E over a given prime number $l\,(\neq p)$. Let E_v denote the completion of E at v, and $\overline{E}_v$ denote an algebraic closure of E_v (which

is isomorphic to $\overline{\mathbb{Q}}_l$). Consider the ring of correspondences A_E with E-coefficients. We have a natural inclusion

$$\iota : A_E \to \mathrm{End}(H^{\bullet}(Y_{\overline{k}}, E_v)),$$

and a natural surjection $A \otimes_{\mathbb{Q}} E \to A_E$, but the injectivity of the last morphism is not clear unless the conjecture $D(Y \times_k Y)$ holds.

As well-known, there exist $R_j(T) \in E[T]$ such that $\pi_j := R_j(g) \in A_E$ is an idempotent, $\sum_j \pi_j = 1$ and the characteristic polynomial of the action of g on $\mathrm{Im}\,\iota(\pi_j)$ is $P_j(T)^{m_j}$. Note that $\mathrm{Im}\,\iota(\pi_j)$ is contained in some cohomology group $H^i(Y_{\overline{k}}, E_v)$ by [4], see [13].

5.2. Proposition. *Let M_j be the motive defined by π_j. Assume $m_j = 1$. Then the endomorphism ring* $\mathrm{End}(M_j)$ *is generated by g over E, and is isomorphic to $E[T]/(P_j(T))$.*

Proof. Let F be a minimal Galois extension of E containing all the roots of $P_j[T]$. Let $d_j = \deg P_j$. For a root α of $P_j[T]$ in F, let $E[\alpha]$ denote the subfield of F generated by α, which is isomorphic to $E[T]/(P_j(T))$. By the same argument as above, we have a projector π_α in $A_{E[\alpha]}$ such that

$$g\pi_\alpha = \alpha\pi_\alpha. \tag{5.2.1}$$

Take any $\Gamma \in A_E$. There exists a polynomial $h(T) \in E[T]$ such that

$$\Gamma\pi_\alpha \cdot \Delta = h(\alpha) \in E[\alpha], \tag{5.2.2}$$

where Δ is the diagonal cycle of Y, and the left-hand side denotes the intersection number. Note that h is independent of α using the action of $\mathrm{Gal}(F/E)$ because, fixing one root β, we have for each root α an automorphism σ of F/E such that $\alpha = \beta^\sigma$. By the Lefschetz trace formula, $h(\alpha)$ coincides with the eigenvalue of the action of Γ on the α-eigenspace of g up to a sign independent of α. Here α denotes also the image of α by the embedding $F \to F_v$ where v is a prime of F over l. So the assertion follows.

5.3. Image of correspondences. Let X be an equidimensional smooth projective k-variety, and $\Gamma \in \mathrm{CH}^r(X \times_k Y)_E$. Assume that E is a subfield of $\mathbb{R}$, $\deg P_j \le 2$ for any j, and the roots of P_j are not real if $\deg P_j = 2$. Let β be an eigenvalue in $\overline{E}_v$ of the Frobenius action on $\mathrm{Im}\,\Gamma_* \cap H^i(Y_{\overline{k}}, E_v)^{\mathrm{prim}}$, and assume that its multiplicity as an eigenvalue of the Frobenius action on $H^i(Y_{\overline{k}}, E_v)^{\mathrm{prim}}$ is 1 (i.e. the dimension of the generalized eigenspace in $H^i(Y_{\overline{k}}, E_v)^{\mathrm{prim}}$ is 1). By definition β is a root of some $P_j(T)$. Assume further that $P_j(T)$ is irreducible over E_v, and $\deg P_j = 2$ (because the case $\deg P_j = 1$ is trivial). Let β' be the other root of $P_j(T)$. Then β' is an eigenvalue of the Frobenius action on $\mathrm{Im}\,\Gamma_* \cap H^i(Y_{\overline{k}}, E_v)^{\mathrm{prim}}$, and hence $\mathrm{Im}\,\Gamma_* \cap H^i(Y_{\overline{k}}, E_v)^{\mathrm{prim}}$ contains $\mathrm{Im}\,\iota(\pi_j)$ on which the canonical pairing is nondegenerate. Indeed, the pairing is compatible with the Frobenius action and has value in $E_v(-i)$ on which the action of g is a multiplication by q^i. So we are interested in the problem: When is P_j irreducible over E_v?

Let F be the field generated by all the roots α of P in $\mathbb{C}$. Since $\alpha\overline{\alpha} = q^r$ with $r \in \mathbb{Z}$, the complex conjugation on F (induced by that on $\mathbb{C}/\mathbb{R}$) is in the center of $\mathrm{Gal}(F/\mathbb{Q})$, because $\alpha^\sigma\overline{\alpha}^\sigma = q^r$ and hence $\overline{\alpha}^\sigma = \overline{\alpha^\sigma}$ for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Let E be the subfield of F fixed by the complex conjugation. By the Tchebotarev density theorem, there are infinitely many

primes v whose density is $1/2$ and such that P_j modulo v is irreducible over the residue field (hence P_j is irreducible over E_v) considering the conjugacy class of the above complex conjugation in $\mathrm{Gal}(F/E)$. So we get the following.

5.4. Proposition. *For prime numbers l such that some v as above is over l, Conjecture* (0.1) *is true if each eigenvalue of the Frobenius action on the intersections of the primitive part with* $\mathrm{Im}\,\rho$ *and with* $\mathrm{Im}\,\gamma$ *has multiplicity* 1 *as an eigenvalue of the Frobenius action on the primitive cohomology $H^j(Y^{(i+1)}, \mathbb{Q}_l)^{\mathrm{prim}}$ for $j \geq 2$ and $i \geq 0$.*

5.5. Remark. If Conjecture (0.1) holds for a general hyperplane section of the generic fiber, then it is enough to consider the intersection with $\mathrm{Im}\,\rho$ by Theorem (0.3). For a general l, Conjecture (0.1) in the case of (5.4) can be reduced to the conjecture D (or B). Thus we can avoid the positivity in this simple case. However, it is not easy to avoid it even in the case where the multiplicity 1 holds for each irreducible component of $Y^{(i+1)}$. Indeed, the ambiguity of the pairing on the motive M_j in (5.2) is given by an automorphism $h \in \mathbb{Q}[g]\,(= \mathbb{Q}[T]/(P_j(T)))$ where $E = \mathbb{Q}$. If there is a morphism of M_j to a direct sum of simple motives which are isomorphic to M_j and indexed by r, and if the morphism to the r-th factor is given by a correspondence Γ_r, then the pull-back of the pairing corresponds to the sum of $h_r := \Gamma'_r \circ \Gamma_r \in \mathbb{Q}[g]$ up to a sign, and the problem is closely related to the standard conjecture of Hodge index type.

References

[1] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.
[2] A.J. de Jong, Smoothness, semi-stability and alternations, Publ. Math. IHES, 83 (1996), 51–93.
[3] P. Deligne, Théorie de Hodge I, Actes Congrès Intern. Math., 1970, vol. 1, 425-430; II, Publ. Math. IHES, 40 (1971), 5–57; III ibid., 44 (1974), 5–77
[4] ———, La conjecture de Weil II, Publ. Math. IHES, 52 (1980), 137–252.
[5] ———, Positivité: signe II (manuscrit, 6–11–85).
[6] ———, Dualité, in SGA 4 1/2, Lect. Notes in Math., vol. 569 Springer, Berlin, 1977, pp. 154–167.
[7] P. Griffiths and J. Harris, On the Noether-Lefschetz theorem and some remarks on codimension-two cycles, Math. Ann. 271 (1985), 31–51.
[8] A. Grothendieck et al, Groupes de monodromie en géométrie algébrique, SGA 7 I, Lect. Notes in Math. vol. 288 Springer, Berlin, 1972.
[9] F. Guillén and V. Navarro Aznar, Sur le théorème local des cycles invariants, Duke Math. J. 61 (1990), 133–155.
[10] R. Hartshorne, Algebraic Geometry, Springer, New York, 1977.
[11] L. Illusie, Autour du théorème de monodromie locale, Astérisque 223 (1994), 9–57.
[12] U. Jannsen, Motives, numerical equivalence, and semi-simplicity, Inv. Math. 107 (1992), 447–452,
[13] N. Katz and W. Messing, Some consequences of the Riemann hypothesis for varieties over finite fields, Inv. Math. 23 (1974), 73–77.

[14] S. Kleiman, Algebraic cycles and Weil conjecture, in Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 359–386.
[15] ———, Motives, in Algebraic Geometry (Oslo, 1970), Wolters-Noordhoff, Groningen, 1972, pp. 53–82.
[16] ———, The standard conjecture, Proc. Symp. Pure Math. 55 (1994), Part 1, 3–20.
[17] S. Lang, Abelian varieties, Interscience Publishers, New York, 1959.
[18] Y.I. Manin, Correspondences, motifs and monoidal transformations, Math. USSR Sb. 6 (1968), 439–470.
[19] J.S. Milne, Etale cohomology, Princeton University Press, 1980.
[20] D. Mumford, Abelian varieties, Oxford University Press, 1970.
[21] M. Rapoport and T. Zink, Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik, Inv. Math. 68 (1982), 21–101.
[22] W. Raskind, Higher l-adic Abel-Jacobi mappings and filtrations on Chow groups, Duke Math. J. 78 (1995), 33–57.
[23] M. Saito, Modules de Hodge polarisables, Publ. RIMS, Kyoto Univ., 24 (1988), 849–995.
[24] M. Saito and S. Zucker, The kernel spectral sequence of vanishing cycles, Duke Math. J. 61 (1990), 329–339.
[25] T. Saito, Weight monodromy conjecture for l-adic representations associated to modular forms, in the Arithmetic and Geometry of Algebraic Cycles, Kluwer Academic, Dordrecht, 2000, pp. 427–431.
[26] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Inv. Math. 22 (1973), 211–319.
[27] A. J. Scholl, Classical Motives, Proc. Symp. Pure Math. 55 (1994), Part 1, 163–187.
[28] B. Segre, Intorno ad teorema di Hodge sulla teoria della base per le curve di una superficie algebrica, Ann. Mat. 16 (1937), 157–163.
[29] J.H.M. Steenbrink, Limits of Hodge structures, Inv. Math. 31 (1975/76), no. 3, 229–257.
[30] J. Tate, Conjectures on algebraic cycles in l-adic cohomology, Proc. Symp. Pure Math. 55 (1994), Part 1, 71–83.
[31] A. Weil, Variétés Abéliennes et Courbes Algébriques, Hermann, Paris, 1948.

Sept. 22, 2003, v.11